EXISTENCE OF NON-ISOTROPIC CONJUGATE POINTS ON RANK ONE NORMAL HOMOGENEOUS SPACES

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Abstract. We give a positive answer to the Chavel's conjecture [J. Diff. Geom. 4 (1970), 13-20]: a simply connected rank one normal homogeneous space is symmetric if any pair of conjugate points are isotropic. It implies that all simply connected rank one normal homogeneous space with the property that the isotropy action is variational complete is a rank one symmetric space.

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1. Introduction

The Jacobi equation for geodesics on a symmetric space has simple solutions and one can directly show that every Jacobi field vanishing at two points is the restriction of a Killing vector field along the geodesic (see for example [12]). A Jacobi field V on a Riemannian manifold (M, g) which is the restriction of a Killing vector field along a geodesic is called *isotropic* [25]. For (M, g) homogeneous Riemannian manifold, it means that V is the restriction of an infinitesimal motion of elements in the Lie algebra of the isometry group $I(M, g)$ of (M, g). Moreover, if V vanishes at a point o of the geodesic then it is obtained as restriction of an infinitesimal K-motion, being K the isotropy subgroup of $I(M, g)$ at $o \in M$. This particular situation was what originally motivated the term "isotropic" (see [6] and [7]).

Two points $p, q \in M$ are said to be *isotropically conjugate* if there exists a non-zero isotropic Jacobi field V along a geodesic passing through p and q such that V vanishes at these points. When every Jacobi field vanishing at p and q is isotropic, we say that they are *strictly isotropic conjugate points*. Then *any pair of conjugate points in a Riemannian symmetric space are strictly isotropic*.

In the case of a naturally reductive space, the adapted *canonical* connection has the same geodesics and the Jacobi equation can be also written as a differential equation with constant coefficients (equation (2.9)). Using this fact, Chavel in [6] and [7] proved that the Berger spaces B^7 and B^{13} admit conjugate points at which no isotropic Jacobi field vanishes. Such spaces are normal homogeneous of rank one, or equivalently, they have positive sectional curvature (see Lemma 2.5), Moreover, in [8], after studding conjugate points on odd-dimensional Berger spheres, he proposed the following conjecture:

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If every conjugate point of a simply connected normal homogeneous Riemannian manifold G/K of rank one is isotropic, then G/K is isometric to a Riemannian symmetric space of rank one.

Here, our main purpose is to prove this conjecture. For it we develop a general theory about the existence of conjugate points which are not isotropic or not strictly isotropic along any geodesic on non-symmetric normal homogeneous spaces (see Theorem 2.9).

The notion of *variationally complete action* was introduced by Bott and Samelson in [5]. They proved that the *isotropy action* on a symmetric space of compact type is variationally complete. In [10] the first named author has proved that if the isotropy action of K on $M = G/K$ is variationally complete then all Jacobi field vanishing at two points is G-isotropic. Then, the above conjecture implies the following:

If the isotropy action on a simply connected rank one normal homogeneous space is variationally complete then it is a compact rank one symmetric space.

Berger [2] has classified the homogeneous spaces G/K which admit a normal G-invariant Riemannian metric with strictly positive sectional curvature. He states that a simply connected, normal homogeneous space of positive curvature is homeomorphic, in fact diffeomorphic (see [15, Proposition 4.3, Ch. II]), to a sphere S^n or one of the projective spaces $\mathbb{C}P^n$, $\mathbb{H}P^n$ or $\mathbb{C}aP^2$, with two exceptions: $B^7 := Sp(2)/SU(2)$ and $B^{13} := SU(5)/H$, where H is a Lie group isomorphic to $(Sp(2) \times S^1)/\pm(id, 1)$. The corresponding $Sp(2)$- and $SU(5)$-standard Riemannian metrics on B^7 and B^{13} have positive sectional curvature. Moreover, the first one is, up to homotheties, the unique $Sp(2)$-invariant Riemannian metric on B^7, since it is isotropy irreducible.

Wilking gives in [24] a new quotient expression $W^7 := (SO(3)\times SU(3))/U^\bullet(2)$ for the positively curved seven-dimensional Aloff-Wallach spaces M^7_{11} [1], equipped with a one-parameter family of bi-invariant metrics on $SO(3) \times SU(3)$, turning them into normal homogeneous spaces. Then, these spaces become a third exception in the Berger's list.

It is worthwhile to note that the above classification is under diffeomorphims and not under isometries. Then, in order to give a proof of the Conjecture, we shall need the following Riemannian classification for this class of spaces, based on results already known (see [23], [25] and [26]) and where δ denotes the corresponding *pinching* constant.

Theorem 1.1. *A simply connected, normal homogeneous space of positive curvature is isometric to one of the following Riemannian spaces:*

(i) *compact rank one symmetric spaces with their standard metrics:* S^n $(\delta = 1)$; $\mathbb{C}P^n$, $\mathbb{H}P^n$, $\mathbb{C}aP^2$ $(\delta = \frac{1}{4})$;
(ii) *the complex projective space* $\mathbb{C}P^n = Sp(m+1)/(Sp(m) \times U(1))$, $n = 2m+1$, *equipped with a standard* $Sp(m+1)$*-homogeneous metric* $(\delta = \frac{1}{16})$.
(iii) *the Berger spheres* $(S^{2m+1} = SU(m+1)/SU(m), g_s)$, $0 < s \leq 1$, $(\delta(s) = \frac{s(m+1)}{8m-3s(m+1)})$.
(iv) $(S^{4m+3} = Sp(m+1)/Sp(m), g_s)$, $0 < s \leq 1$, $(\delta(s) = \frac{s}{8-3s}$, *if* $s \geq \frac{2}{3}$, *and* $\delta(s) = \frac{s^2}{4}$, *if* $s < \frac{2}{3})$.
(v) $B^7 = Sp(2)/SU(2)$ *equipped with a standard* $Sp(2)$*-homogeneous metric.*
(vi) $B^{13} = SU(5)/H$ *equipped with a standard* $SU(5)$*-homogeneous metric.*
(vii) $W^7 = (SO(3)\times SU(3))/U^\bullet(2)$, *equipped with a one-parameter family of* $SO(3)\times SU(3)$*-homogeneous metrics.*

Eliasson [9] and Heintze [14] computed the pinching constants $\frac{1}{37}$ and $\frac{16}{29\cdot 37}$ of B^7 and B^{13} equipped with these normal homogeneous metrics. The pinching constant for any $SU(5)$-invariant metric on B^{13} is obtained by Püttmann [19]. Moreover, he proves that the corresponding optimal pinching constant in B^{13} and also in W^7 is $\frac{1}{37}$.

Up to the Berger space B^7, any non-symmetric rank one normal homogeneous space determines a homogeneous Riemannian fibration over a compact rank one symmetric space. This fact, together with the property that the isotropy action on the unit horizontal tangent sphere at the origin is transitive, simplifies substantially the problem of determining conjugate points along any geodesic in theses spaces. It allows us to show in Theorems 4.8, 4.14 and 4.16 the existence of conjugate points to the origin along any horizontal geodesic starting at this point which are not isotropic. Moreover, for normal homogeneous spaces of type (ii), (iii) and (iv) in Theorem 1.1 we prove that any geodesic admits conjugate points which are not strictly isotropic. Partial results are also given by geodesics in the Berger space B^{13} and the Wilking example W^7.

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2. Normal homogeneous spaces and isotropic Jacobi fields

Let (M,g) be a connected homogeneous *Riemannian* manifold. Then (M,g) can be expressed as coset space G/K, where G is a Lie group, which is supposed to be connected, acting transitively on M, K is the isotropy subgroup of G at some point $o\in M$, the *origin* of G/K, and g is a G-invariant Riemannian metric. Moreover, we can assume that G/K is a *reductive homogeneous space*, i.e., there is an $\mathrm{Ad}(K)$-invariant subspace $\mathfrak{m}$ of the Lie algebra $\mathfrak{g}$ of G such that $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{m}$, $\mathfrak{k}$ being the Lie algebra of K. $(M=G/K,g)$ is said to be *naturally reductive*, or more precisely *G-naturally reductive*, if there exists a reductive decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{m}$ satisfying

$$\langle [X,Y]_{\mathfrak{m}},Z\rangle+\langle [X,Z]_{\mathfrak{m}},Y\rangle=0 \tag{2.1}$$

for all $X,Y,Z\in\mathfrak{m}$, where $[X,Y]_{\mathfrak{m}}$ denotes the $\mathfrak{m}$-component of $[X,Y]$ and $\langle\cdot,\cdot\rangle$ is the $\mathrm{Ad}(K)$-invariant inner product induced by g on $\mathfrak{m}$, by using the canonical identification $\mathfrak{m}\cong T_oM$. When there exists an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$, which we also denote by $\langle\cdot,\cdot\rangle$, whose restriction to $\mathfrak{m}=\mathfrak{k}^{\perp}$ is $\langle\cdot,\cdot\rangle$, the space $(M=G/K,g)$ is called *normal homogeneous*. Then, for all $X,Y,Z\in\mathfrak{g}$, we have

$$\langle [X,Y],Z\rangle+\langle [X,Z],Y\rangle=0. \tag{2.2}$$

Hence each normal homogeneous space is naturally reductive. A G-homogeneous Riemannian manifold is called *standard* if it is normal and $-\langle\cdot,\cdot\rangle$ is the *Killing-Cartan form* of $\mathfrak{g}$. If G is a simple compact Lie group, any naturally reductive G-homogeneous Riemannian manifold is standard, up to scaling factor. Moreover, the unique G-invariant Riemannian metric, up to homotheties, on a compact isotropy irreducible space $M=G/K$ is standard choosing the appropriate scaling factor and this Riemannian metric is Einstein. Notice that not all

standard homogeneous metrics are Einstein and not all normal homogeneous metrics are standard (see [4, Ch. 7] for more details).

Let $\tilde{T}$ denote the torsion tensor and $\tilde{R}$ the corresponding curvature tensor of the *canonical connection* $\tilde{\nabla}$ of (M,g) adapted to the reductive decomposition $\mathfrak{g}=\mathfrak{m}\oplus\mathfrak{k}$ [17, II, p. 190] defined by the sign convention $\tilde{R}(X,Y)=\tilde{\nabla}_{[X,Y]}-[\tilde{\nabla}_X,\tilde{\nabla}_Y]$ and $\tilde{T}(X,Y)=\tilde{\nabla}_XY-\tilde{\nabla}_YX-[X,Y]$, for all $X,Y\in\mathfrak{X}(M)$, the Lie algebra of smooth vector fields on M. Then, these tensor fields at the origin are given by

$$\tilde{T}_o(X,Y)=-[X,Y]_{\mathfrak{m}}\quad,\quad \tilde{R}_o(X,Y)=\mathrm{ad}_{[X,Y]_{\mathfrak{k}}} \tag{2.3}$$

and we have $\tilde{\nabla}g=\tilde{\nabla}\tilde{T}=\tilde{\nabla}\tilde{R}=0$. On naturally reductive homogeneous manifolds $(M=G/K,g)$, the tensor field $S=\nabla-\tilde{\nabla}$, where ∇ denotes the Levi Civita connection of (M,g), is a *homogeneous structure* [21] satisfying $S_XY=-S_YX=-\frac{1}{2}\tilde{T}(X,Y)$, for all $X,Y\in\mathfrak{X}(M)$, and we get

$$\tilde{R}_{XY}=R_{XY}+[S_X,S_Y]-2S_{S_XY}. \tag{2.4}$$

From (2.3) and (2.4), the Riemannian curvature R on a naturally reductive manifold satisfies $<R_{XY}X,Y>=<[[X,Y]_{\mathfrak{k}},X]_{\mathfrak{m}},Y>+\frac{1}{4}\|[X,Y]_{\mathfrak{m}}\|^2$ and, moreover if it is normal, then

$$<R_{XY}X,Y>=\|[X,Y]_{\mathfrak{k}}\|^2+\frac{1}{4}\|[X,Y]_{\mathfrak{m}}\|^2, \tag{2.5}$$

for all $X,Y\in\mathfrak{m}\cong T_oM$. So, the sectional curvature of a normal homogeneous manifold is always non-negative and there exists a section $\pi=\mathbb{R}\{X,Y\}$, $X,Y\in\mathfrak{m}$, such that $K(\pi)=0$ if and only if $[X,Y]=0$.

The notion of Lie triple system given in the theory of symmetric spaces to construct totally geodesic submanifolds can be extended to naturally reductive spaces in the following way [20].

Definition 2.1. Let $(M=G/K,g)$ be a naturally reductive homogeneous manifold with adapted reductive decomposition $\mathfrak{g}=\mathfrak{m}\oplus\mathfrak{k}$. A subspace ν of $\mathfrak{m}$ satisfying $[\nu,\nu]_{\mathfrak{m}}\subset\nu$ and $[[\nu,\nu]_{\mathfrak{k}},\nu]\subset\nu$ is said to be a *Lie triple system* (L.t.s.) of $\mathfrak{m}$.

Lemma 2.2. *If $\nu\subset\mathfrak{m}$ is a L.t.s., then $\mathfrak{g}_\nu=\nu\oplus[\nu,\nu]_{\mathfrak{k}}$ is a Lie subalgebra of $\mathfrak{g}$.*

Proof. First, we need to check the following equality

$$[[X,Y]_{\mathfrak{k}},[Z,W]_{\mathfrak{k}}]=[[X,[Z,W]_{\mathfrak{k}}],Y]_{\mathfrak{k}}+[X,[Y,[Z,W]_{\mathfrak{k}}]_{\mathfrak{k}}, \tag{2.6}$$

for all $X,Y,Z,W\in\mathfrak{m}$. Since $\mathfrak{m}$ is $Ad(K)$-invariant, one obtains $[[X,Y]_{\mathfrak{k}},U]=[[X,Y],U]_{\mathfrak{k}}$, for all $X,Y\in\mathfrak{m}$ and $U\in\mathfrak{k}$. Then, using the Jacobi identity, $[[X,Y]_{\mathfrak{k}},U]=[[X,U],Y]_{\mathfrak{k}}+[X,[Y,U]]_{\mathfrak{k}}$. From here, putting $U=[Z,W]_{\mathfrak{k}}$, we obtain (2.6). Now, taking into account that ν is a L.t.s., $[\nu,\nu]_{\mathfrak{k}}$ is subalgebra of $\mathfrak{k}$ and the result is immediate. □

Denote G_ν the connected Lie subgroup of G with Lie algebra $\mathfrak{g}_\nu$. Then, in similar way than for symmetric spaces (see [15, Theorem 7.2, Ch. IV]), one obtains the following result, based on the fact of that geodesics of M through o are of type $(\exp tu)o$, $u\in\mathfrak{m}$, and the corresponding totally geodesic submanifolds M_ν can be expressed as the orbit $G_\nu\cdot o$ (see [20]).

Proposition 2.3. *Let $(M=G/K,g)$ be a naturally reductive homogeneous manifold with adapted reductive decomposition $\mathfrak{g}=\mathfrak{m}\oplus\mathfrak{k}$ and let $\nu\subset\mathfrak{m}$ be a L.t.s. Then there exists a (unique) complete and connected totally geodesic submanifold M_ν through o such that $T_oM_\nu=\nu$. Moreover, M_ν is the naturally reductive homogeneous manifold $(M_\nu=G_\nu/(G_\nu\cap K),i^*g)$.*

Remark 2.4. We are also interested on M_ν as a closed embedded submanifold of M. These conditions are satisfied when G_ν is a topological subgroup of G and K is compact (see [15, Proposition 4.4, Ch. II]).

From (2.5), a flat totally geodesic submanifold of a normal homogeneous space (M, g) is characterized by the property that the corresponding Lie triple system is an abelian algebra. Defining *rank* of (M, g) as the maximal dimension of a flat, totally geodesic submanifold, one directly obtains

Lemma 2.5. *The following statements are equivalent:*

(i) (M, g) *has positive sectional curvature;*
(ii) $[X, Y] \neq 0$ *for all linearly independent* $X, Y \in \mathfrak{m}$;
(iii) rank $(M, g) = 1$.

For each $X \in \mathfrak{g}$, the mapping $\psi : I\!R \times M \to M$, $(t, p) \in I\!R \times M \mapsto \psi_t(p) = (\exp tX)p$, is a one-parameter group of isometries on $(M = G/K, g)$ and consequently, ψ induces a Killing vector field X^* given by $X^*_p = \frac{d}{dt}_{|t=0}(\exp tX)p$. X^* is called the *fundamental vector field* or the *infinitesimal G-motion* corresponding to X on M. For any $a \in G$, we have

$$(\mathrm{Ad}_a X)^*_{ap} = a_{*p} X^*_p, \tag{2.7}$$

where a_{*p} denotes the differential map of a at $p \in M$.

Definition 2.6. A Jacobi field V along a geodesic γ in $(M = G/K, g)$ is said to be *G-isotropic* if there exists $X \in \mathfrak{g}$ such that $V = X^* \circ \gamma$.

If G is the identity connected component $I_o(M, g)$ of the isometry group $I(M, g)$ of (M, g), then all (complete) Killing vector field on M is a fundamental vector field X^*, for some $X \in \mathfrak{g}$, and we simply say that V is an *isotropic* Jacobi field. Obviously, any G-isotropic Jacobi field is isotropic but the converse does not satisfy in general [12].

From the homogeneity of $M = G/K$, we shall only need to consider geodesics γ emanating from the origin $o \in M$. In what follows, γ_u will denote the unit-speed geodesic starting at o with $\gamma'_u(0) = u \in \mathfrak{m}$, $\|u\| = 1$. Then a Jacobi field V along γ_u with $V(0) = 0$ is G-isotropic if and only if there exists $A \in \mathfrak{k}$ such that $V = A^* \circ \gamma_u$, or equivalently, if there exists an $A \in \mathfrak{k}$ such that [12]

$$(V(0), V'(0)) = (0, [A, u]). \tag{2.8}$$

The linear isotropy representation, i.e. the differential of the action of K on T_oM, corresponds via the natural isomorphism $\pi_* : \mathfrak{m} \to T_oM$ with the Adjoint representation $\mathrm{Ad}(K)$ of K on $\mathfrak{m}$. From (2.7) and since $k \circ \gamma_u = \gamma_{k\cdot u}$, for any $k \in K$, where $k \cdot u = \mathrm{Ad}_k u$, we can state:

Lemma 2.7. *If V is an* (*isotropic*) *Jacobi field along γ_u, then $k_{*\gamma_u}V$ is an* (*isotropic*) *Jacobi field along $\gamma_{k\cdot u}$. Moreover, if $p = \gamma_u(t)$ is a* ((*strictly*) *isotropic*) *conjugate point to the origin along γ_u, then $k(p) = \gamma_{k\cdot u}(t)$ is a* ((*strictly*) *isotropic*) *conjugate point to the origin along $\gamma_{k\cdot u}$.*

On naturally reductive spaces, the connections ∇ and $\tilde{\nabla}$ have the same geodesics and, consequently, the same Jacobi fields (see [25]). Such geodesics can be written as $\gamma_u(t) =$

$(\exp tu)o$ and the Jacobi equation for ∇ coincides with the Jacobi equation for $\tilde{\nabla}$,

$$\frac{\tilde{\nabla}^2 V}{dt^2} - \tilde{T}_{\gamma_u}\frac{\tilde{\nabla} V}{dt} + \tilde{R}_{\gamma_u} V = 0,$$

where $\tilde{R}_{\gamma_u} = \tilde{R}(\gamma_u', \cdot)\gamma_u'$ and $\tilde{T}_{\gamma_u} = \tilde{T}(\gamma_u', \cdot)$. Taking into account that $\tilde{\nabla}\tilde{T} = \tilde{\nabla}\tilde{R} = 0$ and the parallel translation with respect to $\tilde{\nabla}$ of tangent vectors at the origin along γ_u coincides with the differential of $\exp tu \in G$ acting on M, it follows that any Jacobi field V along $\gamma_u(t)$ can be expressed as $V(t) = (\exp tu)_{*o}X(t)$ where $X(t)$ is solution of the differential equation

$$X''(t) - \tilde{T}_u X'(t) + \tilde{R}_u X(t) = 0 \tag{2.9}$$

in the vector space $\mathfrak{m}$, being $\tilde{T}_u X = \tilde{T}(u, X) = -[u, X]_{\mathfrak{m}}$ and $\tilde{R}_u X = \tilde{R}(u, X)u = [[u, X]_{\mathfrak{k}}, u]$ (see [12], [25] for more details).

The following characterization for isotropic Jacobi fields on normal homogeneous spaces is so useful.

Lemma 2.8. [10] *A Jacobi field V along $\gamma_u(t) = (\exp tu)o$ on a normal homogeneous space $(M = G/K, g)$ with $V(0) = 0$ is G-isotropic if and only if $V'(0) \in (\mathrm{Ker}\, \tilde{R}_u)^{\perp}$.*

We end this section establishing a more complete version of [10, Proposition 2.3]. This is a key result for the development of this article.

Theorem 2.9. *Let $(M = G/K, g)$ be a normal homogeneous space and let u, v be orthonormal vectors in $\mathfrak{m}$ verifying $[[u, v], u]_{\mathfrak{m}} = \lambda v$, for some $\lambda > 0$.*

- (i) *If $[u, v]_{\mathfrak{m}} = 0$, then $\gamma_u(\frac{p\pi}{\sqrt{\lambda}})$, for $p \in \mathbb{Z}$, are G-isotropically conjugate points to the origin.*
- (ii) *If $[u, v] \in \mathfrak{m}\backslash\{0\}$ and $[[u, [u, v]]_{\mathfrak{k}}, u] = \rho[u, v]$, then $\rho = \frac{1}{\lambda}\|[[u, v], u]_{\mathfrak{k}}\|^2$ and the following cases hold:*
  - (A) *If $\rho = 0$, i.e. $[[u, v], u] \in \mathfrak{m}$, $\gamma_u(\frac{2p\pi}{\sqrt{\lambda}})$, $p \in \mathbb{Z}$, are conjugate points to the origin but not strictly G-isotropic.*
  - (B) *If $\rho > 0$, $\gamma_u(\frac{s}{\sqrt{\lambda+\rho}})$, where*
    1. *s is a solution of the equation $\tan\frac{s}{2} = -\frac{\rho s}{2\lambda}$, or*
    2. *$s = 2p\pi$, $p \in \mathbb{Z}$,*

    are conjugate points to the origin along $\gamma_u(t) = (\exp tu)o$. In the first case, they are not strictly G-isotropic and in the second one, they are G-isotropic.

Proof. (i) From (2.3) we get $\tilde{T}_u v = 0$ and $\tilde{R}_u v = \lambda v$. Then, $X(t) = A\sin\sqrt{\lambda}tv$ is a solution of (2.9) with $X(0) = 0$. Because from (2.2) $\tilde{R}_u$ is self-adjoint, $v \in (\mathrm{Ker}\, \tilde{R}_u)^{\perp}$ and using Lemma 2.8, $V(t) = (\exp tu)_{*o}X(t)$ is G-isotropic.
(ii) Here, we obtain

$$\begin{array}{rclcrcl} \tilde{T}_u v & = & -\sqrt{\lambda}w, & \quad & \tilde{T}_u w & = & \sqrt{\lambda}v, \\ \tilde{R}_u v & = & 0, & & \tilde{R}_u w & = & \rho w, \end{array}$$

where $w = \frac{1}{\sqrt{\lambda}}[u, v]$. From (2.1) and (2.2) it follows that $\lambda = \|[u, v]\|^2$ and $\rho = \|[u, w]_{\mathfrak{k}}\|^2$, which implies that $\rho = <\tilde{R}_u w, w>$. Hence, the solutions $X(t) = X^1(t)v + X^2(t)w$ of (2.9) satisfy

$$\left\{ \begin{array}{l} X^{1\prime\prime} - \sqrt{\lambda}X^{2\prime} = 0, \\ X^{2\prime\prime} + \sqrt{\lambda}X^{1\prime} + \rho X^2 = 0. \end{array} \right.$$

Now, differentiating the second equation and substituting ${X^1}''$ from the first one, we get

$$X^{2'''} + (\lambda+\rho)X^{2'} = 0.$$

Therefore, $X(t)$ with $X(0)=0$ is given by

$$(2.10)\qquad X(t) = \sqrt{\frac{\lambda}{\lambda+\rho}}\Big(A(1-\cos\sqrt{\lambda+\rho}t) - B(\frac{\rho\sqrt{\lambda+\rho}}{\lambda}t + \sin\sqrt{\lambda+\rho}t)\Big)v + \Big(A\sin\sqrt{\lambda+\rho}t + B(1-\cos\sqrt{\lambda+\rho}t)\Big)w,$$

where A, B are constant. Then $X'(0) = -B\frac{\lambda+\rho}{\sqrt{\lambda}}v + A\sqrt{\lambda+\rho}w$ and from Lemma 2.8, the Jacobi vector fields $V(t) = (\exp tu)_{*o})X(t)$ along γ_u with $B \neq 0$ are not G-isotropic. Hence, if $\rho = 0$, one gets $X(\frac{2p\pi}{\sqrt{\lambda}}) = 0$, for $p \in \mathbb{Z}$, and it proves (A).

Next, suppose $\rho > 0$. The values of t such that $X(t) = 0$, $t \neq 0$, for some $A, B \in I\!R$, are the zeros of the determinant

$$\begin{vmatrix} 1-\cos\sqrt{\lambda+\rho}t & -(\frac{\rho\sqrt{\lambda+\rho}}{\lambda}t + \sin\sqrt{\lambda+\rho}t) \\ \sin\sqrt{\lambda+\rho}t & 1-\cos\sqrt{\lambda+\rho}t \end{vmatrix},$$

that is, the zeros of the function $f(s) = 1-\cos s - \mu s\sin s$, where $s = s(t) = \sqrt{\lambda+\rho}t$ and $\mu = -\frac{\rho}{2\lambda}$. Hence, it follows that either $\sin s = 0$, which gives $s = 2p\pi$, $p \in \mathbb{Z}$, or,

$$\cos s = \frac{1-\mu^2 s^2}{1+\mu^2 s^2}, \qquad \sin s = \frac{2\mu s}{1+\mu^2 s^2},$$

which yields to the equation $\tan\frac{s}{2} = \mu s$. Let c be a solution of $\tan\frac{s}{2} = \mu s$. Then $f(c) = 0$ and it implies, substituting in (2.10), that $A = -\mu c B$. So the vector fields $V(t)$ along γ_u given by

$$V(t) = (\exp tu)_{*o}\Big(\sqrt{\tfrac{\lambda}{\lambda+\rho}}\Big(\mu c(1-\cos s(t)) + (2\mu s(t) + \sin s(t))\Big)v + (\mu c\sin s(t) - (1-\cos s(t)))w\Big),$$

are Jacobi fields such that $V(\frac{c}{\sqrt{\lambda+\rho}}) = 0$ and they are not G-isotropic.

Finally, because $\tilde{R}_u w = \rho w$ one gets that $w \in (\mathrm{K}er\ \tilde{R}_u)^{\perp}$ and hence, from Lemma 2.8, the vector fields along γ_u spanned by

$$V(t) = (\exp tu)_{*o}\Big(\sqrt{\frac{\lambda}{\lambda+\rho}}(1-\cos s)v + \sin s w\Big)$$

are G-isotropic Jacobi fields with $V(\frac{2p\pi}{\sqrt{\lambda+\rho}}) = 0$, for all $p \in \mathbb{Z}$. □

Remark 2.10. The equation $\tan\frac{s}{2} = -\frac{\rho s}{2\lambda}$ has for $\rho > 0$ a solution $s_o \in]\pi, 2\pi[$. Therefore, $\gamma_u(\frac{s_o}{\sqrt{\lambda+\rho}})$ is the first conjugate point of γ_u to the origin among those here obtained and then the conjugate radius is $\leq \frac{s_o}{\sqrt{\lambda+\rho}}$.

3. Normal homogeneous metrics of positive curvature on symmetric spaces

A homogeneous Riemannian manifold with positive sectional curvature is compact and if it is moreover simply connected, it can be written as G/K with G a compact Lie group. Wallach [23, Theorem 6.1] showed that for the even-dimensional case, $\dim M = 2n$, it is isometric to

(i) a compact rank one symmetric space: $\mathbb{C}P^n$, S^{2n}, $\mathbb{H}P^{n/2}$ (n even), $\mathbb{C}aP^2$ ($n=8$); or

(ii) one of the following quotient spaces G/K with a suitable G-invariant metric:

(1) the manifolds of flags $\mathbb{F}^6$, $\mathbb{F}^{12}$ and $\mathbb{F}^{24}$ in $\mathbb{C}P^2$, $\mathbb{H}P^2$ and $\mathbb{C}aP^2$, respectively:

$$\begin{array}{rcl} \mathbb{F}^6 & = & SU(3)/(S(U(1)\times U(1)\times U(1)), \\ \mathbb{F}^{12} & = & Sp(3)/(SU(2)\times SU(2)\times SU(2)), \\ \mathbb{F}^{24} & = & F_4/\,Spin(8); \end{array}$$

(2) $\mathbb{C}P^n = Sp(m+1)/(Sp(m)\times U(1))$, $n = 2m+1$;

(3) the six-dimensional sphere $S^6 = G_2\,/\,SU(3)$.

Remark 3.1. In (i), the quotient spaces determined by the pairs $(SU(n+1), S(U(n)\times U(1)))$, $(Spin(2n+1), Spin(2n))$, $(Sp(n), Sp(n-1)\times Sp(1))$ and $(F_4, Spin(9))$ are isotropy-irreducible and so, they admit, up to homotheties, a unique invariant Riemannian metric.

Valiev [22] determined the set of all homogeneous Riemannian metrics on $\mathbb{F}^6$, $\mathbb{F}^{12}$ and $\mathbb{F}^{24}$ of positive sectional curvature and their corresponding optimal pinching constants. (Pinching constants means how much the local geometry of a compact Riemannian manifold (M,g) with positive sectional curvature K deviates from the geometry of a standard sphere. They are defined as quotients $\delta(M,g) = \frac{\min K}{\max K}$ of the extremal values of the sectional curvature.) According to the Berger's classification, they cannot be normal homogeneous. It is worthwhile to note that $S^6 = G_2\,/\,SU(3)$ carries the usual metric of constant sectional curvature [3], the isotropy action of $SU(3)$ is irreducible on the tangent space (see [4]) and it is a nearly-Kähler 3-symmetric space but $(G_2, SU(3))$ is not a symmetric pair.

The complex projective space $\mathbb{C}P^n$, $n = 2m+1$, equipped with the standard $Sp(m+1)$-homogeneous Riemannian manifold, is also an irreducible compact nearly Kähler 3-symmetric space [13]. It can be viewed as the base space of the Hopf fibration

$$S^1 \to S^{4m+3} = Sp(m+1)/\,Sp(m) \to \mathbb{C}P^n = Sp(m+1)/(Sp(m)\times U(1)).$$

Here, the inclusion of $Sp(m)$ in $Sp(m+1)$ is the standard one. Taking $Sp(m+1)$ as Lie subgroup of $SU(2(m+1))$ also in the natural way, we obtain a reductive decomposition $\mathfrak{sp}(m+1) = \mathfrak{sp}(m)\oplus\mathfrak{m}$ adapted to the quotient $Sp(m+1)/\,Sp(m)$, with $\mathfrak{m} = \mathfrak{m}_0\oplus\mathfrak{m}_1$, being $\mathfrak{m}_0 = \mathfrak{su}(2)\cong\mathfrak{sp}(1)$ and $\mathfrak{m}_1\cong\mathbb{H}^m$. The isotropy Lie subgroup $Sp(m)$ acts trivially on $\mathfrak{m}_0$ and by its standard representation on $\mathfrak{m}_1$.

Let E_{ij} denote the square matrix on $\mathfrak{su}(2(m+1))$ with entry 1 where the ith row and the jth column meet, all other entries being 0, and set

$$\begin{array}{rcl} A_{jk} & = & \sqrt{-1}(E_{jj}-E_{kk}), \\ B_{jk} & = & E_{jk}-E_{kj}, \\ C_{jk} & = & \sqrt{-1}(E_{jk}+E_{kj}). \end{array} \tag{3.11}$$

In what follows we shall need the Lie multiplication table:

$$
\begin{array}{rcl}
[A_{rj},A_{kl}] &=& 0,\\
[A_{rj},B_{kl}] &=& \delta_{rk}C_{rl}-\delta_{rl}C_{rk}-\delta_{jk}C_{jl}+\delta_{jl}C_{jk},\\
[A_{rj},C_{kl}] &=& -\delta_{rk}B_{rl}-\delta_{rl}B_{rk}+\delta_{jk}B_{jl}+\delta_{jl}B_{jk},\\
[B_{rj},B_{kl}] &=& \delta_{jk}B_{rl}-\delta_{jl}B_{rk}-\delta_{rk}B_{jl}+\delta_{rl}B_{jk},\\
[B_{rj},C_{kl}] &=& \delta_{jl}C_{rk}+\delta_{jk}C_{rl}-\delta_{rl}C_{jk}-\delta_{rk}C_{jl},\\
[C_{rj},C_{kl}] &=& -\delta_{jk}B_{rl}-\delta_{jl}B_{rk}-\delta_{rk}B_{jl}-\delta_{rl}B_{jk}.
\end{array}
\tag{3.12}
$$

The canonical basis $Y_1,\dots,Y_m$ of $\mathbb{H}^m$ over $\mathbb{H}$ is given by $Y_\alpha = B_{\alpha,2m+1}+B_{m+\alpha,2(m+1)}$, for $\alpha=1,\dots,m$. If we put $X_1 = i = A_{2m+1,2(m+1)}$, $X_2 = j = B_{2m+1,2(m+1)}$ and $X_3 = k = C_{2m+1,2(m+1)}$ as basis of $\mathfrak{m}_0$ then the corresponding basis $\{Y_\alpha;Y_{\alpha 1};Y_{\alpha 2};Y_{\alpha 3}\}_{\alpha=1}^m$ over $I\!R$, where $Y_{\alpha 1}=iY_\alpha$, $Y_{\alpha 2}=jY_\alpha$, $Y_{\alpha 3}=kY_\alpha$ is obtained by using of Lie brackets on $\mathfrak{m}$. For p,q,r a cyclic permutation of $1,2,3$, one gets:

$$
\begin{array}{rclrclrcl}
[X_p,X_q] &=& 2X_r, & [X_p,Y_\alpha] &=& -Y_{\alpha p}, & [X_p,Y_{\alpha p}] &=& Y_\alpha,\\
[X_p,Y_{\alpha q}] &=& Y_{\alpha r}, & [Y_\alpha,Y_\beta] &=& -Z_{\alpha,\beta}, & [Y_\alpha,Y_{\alpha p}] &=& -2X_p+2Z_{\alpha p},\\
[Y_{\alpha p},Y_{\alpha q}] &=& 2X_r+2Z_{\alpha r}, &&&&&&
\end{array}
\tag{3.13}
$$

where $Z_{\alpha,\beta}=B_{\alpha,\beta}+B_{m+\alpha,m+\beta}$, $1\le\alpha<\beta\le m$, and $Z_{\alpha 1}=A_{\alpha,m+\alpha}$, $Z_{\alpha 2}=B_{\alpha,m+\alpha}$ and $Z_{\alpha 3}=C_{\alpha,m+\alpha}$, $1\le\alpha\le m$. Moreover, for $\alpha\ne\beta$,

$$
[Y_\alpha,Y_{\beta p}]=Z_{(\alpha,\beta)p},\qquad [Y_{\alpha p},Y_{\beta p}]=-Z_{\alpha,\beta},\qquad [Y_{\alpha p},Y_{\beta q}]=Z_{(\alpha,\beta)r},
\tag{3.14}
$$

where $Z_{(\alpha,\beta)1}=C_{\alpha,\beta}-C_{m+\alpha,m+\beta}$, $Z_{(\alpha,\beta)2}=B_{\alpha,m+\beta}+B_{\beta,m+\alpha}$ and $Z_{(\alpha,\beta)3}=C_{\alpha,m+\beta}+C_{m+\alpha,\beta}$.

A reductive decomposition of $\mathfrak{sp}(m+1)$ adapted to $Sp(m+1)/(Sp(m)\times U(1))$ is then $\mathfrak{sp}(m+1)=(\mathfrak{sp}(m)\oplus\mathfrak{u}(1))\oplus\mathfrak{p}$, where $\mathfrak{u}(1)$ is generated by X_1 and $\mathfrak{p}=\mathfrak{p}_0\oplus\mathfrak{p}_1$, being $\mathfrak{p}_0$ the subspace generated by X_2 and X_3 and $\mathfrak{p}_1=\mathfrak{m}_1\cong\mathbb{H}^m$. In fact, $\mathfrak{p}_0$ is a Lie triple system with corresponding totally geodesic submanifold through the origin the 2-sphere $Sp(1)/U(1)$. Hence, one obtains the *homogeneous fibration* over $\mathbb{H}P^m$

$$
Sp(1)/U(1)\to Sp(m+1)/(Sp(m)\times U(1))\to Sp(m+1)/(Sp(m)\times Sp(1)).
$$

Every $Sp(m+1)$-invariant metric on $\mathbb{C}P^n = Sp(m+1)/(Sp(m)\times U(1))$ is determined by an inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{p}$ such that $\{Y_\alpha;iY_\alpha;jY_\alpha;kY_\alpha\}_{\alpha=1}^m$ is an orthonormal basis of $\mathfrak{p}_1$, $\mathfrak{p}_0$ is orthogonal to $\mathfrak{p}_1$ and $\|X_2\|^2=\|X_3\|^2=s$, for some $s>0$. Then, if we suppose that $\langle\cdot,\cdot\rangle$ is $\mathrm{Ad}(Sp(m+1))$-invariant, it follows, using (2.2) and (3.13), that

$$
1=\langle jY_\alpha,jY_\alpha\rangle=\langle[Y_\alpha,X_2],jY_\alpha\rangle=-\langle[Y_\alpha,jY_\alpha],X_2\rangle=2\langle X_2,X_2\rangle=2s.
$$

Hence, $s=\frac{1}{2}$ and then $\langle X,Y\rangle=-\frac{1}{4}\mathrm{trace}\,XY$, for all $X,Y\in\mathfrak{sp}(m+1)$. It determines a standard $Sp(m+1)$-homogeneous Riemannian metric with positive sectional curvature and pinching $\delta=\frac{1}{16}$ [26] and so, it is not the symmetric Fubini-Study one. It proves the following:

Proposition 3.2. *A simply connected, $2n$-dimensional, normal homogeneous space of positive sectional curvature is isometric to a compact rank one symmetric space: S^{2n} ($\delta=1$); $\mathbb{C}P^n$, $\mathbb{H}P^{n/2}$ (n even), $\mathbb{C}aP^2$ ($n=8$), ($\delta=\frac{1}{4}$); or to the complex projective space $\mathbb{C}P^n = Sp(m+1)/(Sp(m)\times U(1))$, $n=2m+1$, equipped with the standard $Sp(m+1)$-homogeneous Riemannian metric ($\delta=\frac{1}{16}$).*

Remark 3.3. The normal $Sp(m+1)$-homogeneous metric on $\mathbb{C}P^n$ is Einstein if and only if $n = 3$ [26].

Using the list of groups acting transitively on spheres given by Montgomery, Samelson and Borel (see [18]), Ziller [26] obtained all homogeneous Riemannian metrics on the sphere S^n. Next, we find those which are normal homogeneous with positive curvature.

Proposition 3.4. *All normal homogeneous metrics on spheres have positive sectional curvature and they determine the following Riemannian manifolds:*

(i) *the Euclidean sphere* S^n;
(ii) $(S^{2m+1} = SU(m+1)/\,SU(m), g_s)$;
(iii) $(S^{4m+3} = Sp(m+1)/\,Sp(m), g_s)$,

where in (ii) *and* (iii) g_s, $0 < s \leq 1$, *denotes a normal homogeneous metric with corresponding constant pinching* $\delta(s) = \frac{s(m+1)}{8m-3s(m+1)}$ *and*

$$\delta(s) = \begin{cases} \frac{s}{8-3s} & \text{if } s \in [\frac{2}{3}, 1]; \\ \frac{s^2}{4}, & \text{if } s \in]0, \frac{2}{3}[. \end{cases}$$

In (ii) *for* $s < 1$ $(s = 1)$ *they are* $SU(m+1) \times U(1)$ $(SU(m+1))$*-normal homogeneous and in* (iii)*,* $Sp(m+1) \times Sp(1)$ $(Sp(m+1))$*-normal homogeneous.*

Proof. Consider the complex projective space $\mathbb{C}P^m = SU(m+1)/S(U(m) \times U(1))$ as Hermitian symmetric space, equipped with the $SU(m+1)$-standard homogeneous metric determined by $\langle X, Y\rangle = -\frac{1}{2}\text{trace } XY$ on $\mathfrak{su}(m+1)$. Let $\mathfrak{su}(m+1) = \mathfrak{k} \oplus \mathfrak{m}_1$ be the canonical decomposition, where $\mathfrak{k}$ is the Lie algebra of the isotropy group and $\mathfrak{m}_1 = \mathfrak{k}^\perp \cong \mathbb{C}^m$. The center $\mathfrak{Z}(\mathfrak{k})$ of $\mathfrak{k}$ is one-dimensional and we have $\mathfrak{k} = \mathfrak{su}(m) \oplus \mathfrak{Z}(\mathfrak{k})$. Let $Z_0 \in \mathfrak{Z}(\mathfrak{k})$ such that $\|Z_0\| = 1$. Then Z_0 determines a unit Killing vector field ξ on $S^{2m+1} = SU(m+1)/\,SU(m)$ and its integral curves -they are geodesics- are the fibers of the Hopf fibration $S^1 \to S^{2m+1} \to \mathbb{C}P^m$. The natural projection is a Riemannian submersion, where S^{2m+1} is considered with the $SU(m+1)$-standard metric. Then all $SU(m+1)$-invariant metrics on S^{2m+1} are obtained, up to a scaling factor, taking $\langle \cdot, \cdot\rangle$ on $\mathfrak{m}_1$, Z_0 orthogonal to $\mathfrak{m}_1$ and $\|Z_0\|^2 = s$, for same $s > 0$. From [25, Theorem 3], each one of these metrics g_s is $(SU(m+1) \times U(1))$-naturally reductive; for $s \neq 1$ it is not $SU(m+1)$-normal homogeneous and g_s is $(SU(m+1) \times U(1))$-normal homogeneous if and only if $s < 1$. Then the metrics g_s, for $s \leq 1$, are normal homogeneous. Moreover, they have positive sectional curvature. The minimum and maximum sectional curvatures are [25]: $s\frac{m+1}{2m}$ and $4 - 3s\frac{m+1}{2m}$ and it gives the corresponding pinching constants $\delta(s)$.

Every $Sp(m+1)$-invariant metric on S^{4m+3} is determined, up to scaling factor, by an inner product $\langle \cdot, \cdot\rangle$ on $\mathfrak{m}$ such that $\{Y_\alpha; iY_\alpha; jY_\alpha; kY_\alpha\}_{\alpha=1}^n$ is an orthonormal basis of $\mathfrak{m}_1$, $\mathfrak{m}_0$ is orthogonal to $\mathfrak{m}_1$ and $\|X_p\|^2 = t_p$, for some $t_p > 0$, $p = 1, 2, 3$. From (3.13), $\langle \cdot, \cdot\rangle$ is naturally reductive if and only if $t_1 = t_2 = t_3 = \frac{1}{2}$. Because $Sp(m)$ acts trivially on $\mathfrak{m}_0$ and $[\mathfrak{m}_0, \mathfrak{m}_0] \subset \mathfrak{m}_0$, it follows from [25, Theorem 3] that the $Sp(m+1)$-invariant metrics g_s on S^{4m+3} where $t_1 = t_2 = t_3 = \frac{s}{2}$ are $Sp(m+1) \times Sp(1)$-naturally reductive and $Sp(m+1) \times Sp(1)$-normal homogeneous if and only if $s < 1$. So the metrics g_s, for $s \leq 1$, are normal homogeneous. From [26], they have positive sectional curvature with corresponding pinching $\delta(s)$ as in (iii).

The rest of Lie groups acting transitively on spheres and with reducible isotropy are $Sp(m+1)\times U(1)$ on $S^{4m+3} = Sp(m+1)\times U(1)/\,Sp(m)\times U(1)$ and $Spin(9)$ on $S^{15} = Spin(9)/\,Spin(7)$. For both cases, all invariant metrics, different from those given above, are not naturally reductive [26]. Those with irreducible isotropy are $SO(n+1)$, acting on $S^n = SO(n+1)/\,SO(n)$, $Spin(7)$ on $S^7 = Spin(7)/\,G_2$ and G_2 on $S^6 = G_2\,/\,SU(3)$. In all three cases the unique invariant metric, up to a scalar factor, has constant sectional curvature [3]. Hence, it follows that all normal homogeneous metrics on spheres have positive sectional curvature and it gives the result. $\square$

Remark 3.5. S^{2m+1} equipped with one of the metrics g_s, $s \leq 1$, in (ii) is known as a *Berger sphere*. Such metrics are not Einstein except the one of constant curvature for $m = 1$ [16]. Since $U(m+1)$ also acts by isometries on (S^{2m+1}, g_s), it follows that the set of $U(m+1)$-invariant metrics on S^{2m+1} coincides with the set of $SU(m+1)$-invariant metrics g_s.

Following [26], the unique non-Euclidean normal homogeneous Riemannian metric on spheres which is Einstein is given in (iii) for $s = \frac{2}{2m+3}$ on S^{4m+3}, which is the Jensen's example [16].

4. Existence of non-isotropic Jacobi fields

Let $(M = G\,/\,K, g)$ be a normal homogeneous space and $\langle\cdot,\cdot\rangle$ its corresponding bi-invariant inner product on the Lie algebra $\mathfrak{g}$ of G. Consider a closed subgroup H of G such that $K \subset H \subset G$. Then we define the homogeneous fibration

$$F = H\,/\,K \to M = G\,/\,K \to \tilde{M} = G\,/\,H \quad : \quad g\,K \mapsto g\,H,$$

with fiber F and structural group H. The $\langle\cdot,\cdot\rangle$-orthogonal decompositions $\mathfrak{h} = \mathfrak{k}\oplus\mathfrak{m}_0$, $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{m} = \mathfrak{k}\oplus\mathfrak{m}_0\oplus\mathfrak{m}_1$ and $\mathfrak{g} = \mathfrak{h}\oplus\mathfrak{m}_1$ determine reductive decompositions for F, M and $\tilde{M}$, respectively, and $\mathfrak{m}_0$ and $\mathfrak{m}_1$ are invariant under the linear isotropy action of K. The projection $\pi : (M, g) \to (\tilde{M}, \tilde{g})$, where $\tilde{g}$ is induced by $\langle\cdot,\cdot\rangle_{\mathfrak{m}_1\times\mathfrak{m}_1}$, is a Riemannian submersion and, because $\mathfrak{m}_0$ is a L.t.s. of $\mathfrak{m}$, its fibers are totally geodesic.

Given a horizontal geodesic $\gamma_u(t) = (\exp tu)o$, $u \in \mathfrak{m}_1$, of (M, g), we consider the geodesic $\tilde{\gamma}_u(t) = (\pi\circ\gamma_u)(t) = (\exp tu)\tilde{o}$ of $(\tilde{M}, \tilde{g})$, where $\tilde{o}$ is the origin of $\tilde{M}$.

Lemma 4.1. *If $\gamma_u(t_0)$ is a G-isotropic conjugate point to o in M, then $\tilde{\gamma}_u(t_0)$ is G-isotropic conjugate to $\tilde{o}$ in $\tilde{M}$.*

Proof. Let V be a non-zero G-isotropic Jacobi field along γ_u with $V(0) = 0$. Then there exists $A \in \mathfrak{k}$ such that $V = A^*\circ\gamma_u$. Putting $\tilde{V} = \pi_{*\gamma_u}V$, one gets

$$\tilde{V}(t) = \pi_{*\gamma_u(t)}\frac{d}{ds}_{|s=0}(\exp sA)\gamma_u(t) = \frac{d}{ds}_{|s=0}\pi((\exp sA)\gamma_u(t)) = \frac{d}{ds}_{|s=0}(\exp sA)\tilde{\gamma}_u(t) = A^*_{\tilde{\gamma}_u}.$$

Hence, $\tilde{V}$ is a G-isotropic Jacobi field along $\tilde{\gamma}_u$, which using (2.8) is not identically zero and if $V(t_0) = 0$, for some t_0, then $\tilde{V}(t_0) = 0$. $\square$

Let $\mathcal{S} \subset \mathfrak{m}$ be the unit sphere of $\mathfrak{m} = \mathfrak{k}^\perp$ and let $\mathfrak{n} \subset \mathfrak{m}$ be an $\mathrm{Ad}(K)$-invariant subspace of $\mathfrak{m}$. Since $\langle\cdot,\cdot\rangle$ is $\mathrm{Ad}(K)$-invariant, the linear isotropy action can be restricted to $\mathcal{S}\cap\mathfrak{n}$.

Proposition 4.2. *The following conditions are equivalent:*

(i) *the linear isotropy action of* K *on* $\mathcal{S}\cap\mathfrak{n}$ *is transitive;*
(ii) $[u,v]_{\mathfrak{k}}\neq 0$ *for all linearly independent* $u,v\in\mathfrak{n}$;
(iii) *there exists* $u\in\mathfrak{n}\setminus\{0\}$ *such that* $[u,v]_{\mathfrak{k}}\neq 0$ *for all* $v\in\mathfrak{n}$ *linearly independent to* u.

Proof. Because $\frac{d}{dt}_{|t=0}\mathrm{Ad}_{\exp tZ}u=\frac{d}{dt}_{|t=0}(e^{t\mathrm{ad}Z})u=[Z,u]$, for all $Z\in\mathfrak{k}$ and $u\in\mathfrak{m}$, it follows that the tangent space $T_u(K\cdot u)$ of the orbit $K\cdot u=\mathrm{A}d(K)u$ is the bracket $[\mathfrak{k},u]$. Hence the action $K\times(\mathcal{S}\cap\mathfrak{n})\to\mathcal{S}\cap\mathfrak{n}$ is transitive if and only if, for some $u\in\mathcal{S}\cap\mathfrak{n}$, the orthogonal complement $\mathfrak{n}_1$ of $[\mathfrak{k},u]$ in $\mathfrak{n}$ is generated by u. But from (2.2) we have $\mathfrak{n}_1=\{v\in\mathfrak{n}\mid\langle[\mathfrak{k},u],v\rangle=0\}=\{v\in\mathfrak{n}\mid\langle[u,v],\mathfrak{k}\rangle=0\}=\{v\in\mathfrak{n}\mid[u,v]_{\mathfrak{k}}=0\}$ and it proves the result. □

As a direct consequence, using Lemma 2.5, one obtains the following well-known result.

Corollary 4.3. *Let* $M=G/K$ *be a symmetric space,* (G,K) *a Riemannian symmetric pair and* G *semi-simple. Then the linear isotropy action of* K *on the unit sphere of* $\mathfrak{m}$ *is transitive if and only if* $\operatorname{rank} M=1$.

Each unit vector $u\in\mathfrak{m}=\mathfrak{m}_0\oplus\mathfrak{m}_1$ can be written as

$$u=u(\theta)=\cos\theta u_0+\sin\theta u_1, \tag{4.15}$$

where $u_0\in\mathfrak{m}_0$ and $u_1\in\mathfrak{m}_1$, $\|u_0\|=\|u_1\|=1$, and θ is the slope angle $\theta=\mathrm{a}ng(u,\mathfrak{m}_0)\in[0,\pi/2]$. From Lemma 2.7, if the linear isotropy action of K on $\mathcal{S}\cap\mathfrak{m}_1$ is transitive the study of conjugate points along geodesics on normal homogeneous spaces $(M=G/K,g)$ can be reduced to consider geodesics γ_u where u is given as in (4.15) but fixing an arbitrary unit vector $u_1\in\mathfrak{m}_1$. Using this fact and Theorem 2.9, together with Lemma 4.1, we shall find isotropic and non-isotropic conjugate points to the origin in geodesics starting at that point.

4.1. $(S^{2m+1},g_{\kappa,s})$, $(S^{4m+3},g_{\kappa,s})$, $(s<1)$, AND $(\mathbb{C}P^{2m+1},g_\kappa)$.

Denote by $S^m(\kappa)$ the sphere of radius $\frac{1}{\sqrt{\kappa}}$, or equivalently, of constant curvature $\kappa>0$, by $I\!\!RP^m(\kappa)$ the real projective space of constant curvature κ, by $\mathbb{C}P^m(\kappa)$ the complex projective space with constant holomorphic sectional curvature $c=4\kappa$ and by $\mathbb{H}P^m(\kappa)$ the quaternionic projective space with constant quaternionic sectional curvature $c=4\kappa$ (the sectional curvature varies between κ and 4κ).

In order to study conjugate points along geodesics in $(S^{2m+1},g_{\kappa,s}=\frac{1}{\kappa}g_s)$, $(S^{4m+3},g_{\kappa,s}=\frac{1}{\kappa}g_s)$ and $(\mathbb{C}P^{2m+1},g_\kappa=\frac{1}{\kappa}g)$, we first need to determine the reductive decompositions $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{m}_s$ (they only depend on the parameter s) associated to the corresponding normal homogeneous quotients. For $s=1$, including in this case $(\mathbb{C}P^{2m+1},g_\kappa)$, such reductive decompositions have already been obtained in the previous section.

We start considering Berger spheres $(S^{2m+1}=SU(m+1)/SU(m),g_s)$, $s<1$. Put $S_j=\frac{1}{\alpha_j}\sum_{l=1}^{j}lA_{l,l+1}$, $j=1,\dots,m$, where $\alpha_j=(\frac{j(j+1)}{2})^{1/2}$. Then an orthonormal basis of $\mathfrak{su}(m+1)$ with respect to the inner product $<X,Y>=-\frac{1}{2}\mathrm{trace}\,XY$, is given by $\{A_{l,l+1},\ l=1,\dots,m;\ B_{rj},\ C_{rj};\ 1\le r<j\le m+1\}$ and $\{S_j,j=1,\dots,m-1;\ B_{rj},\ C_{rj};\ 1\le r<j\le m\}$ is an orthonormal basis of $\mathfrak{su}(m)$ embedded in the usual way in $\mathfrak{su}(m+1)$. Moreover, up to sign, $Z_0=S_m$, where Z_0 is defined as in proof of Proposition 3.4. An orthonormal basis for $\mathfrak{m}_1\cong\mathbb{C}^m$ is given by

$$e_r=B_{r,m+1},\quad f_r=C_{r,m+1},\quad r=1,\dots,m.$$

Let D be a basis element of the Lie algebra $I\!R$ of $S^1 \cong U(1)$ and consider the coset space G/K, where $G = SU(m+1) \times U(1)$ and K is the connected Lie subgroup of G with Lie algebra $\mathfrak{k} = \mathfrak{su}(m) \oplus I\!R$ generated by

$$\{h_s = \sqrt{1-s}(Z_0 + D), S_1, \ldots, S_{m-1}; B_{rj}, C_{rj}, 1 \leq r < j \leq m\}.$$

Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}_s$ is a reductive decomposition for G/K being $\mathfrak{m}_s = \mathfrak{m}_{0s} \oplus \mathfrak{m}_1$, $\mathfrak{m}_{0s} = I\!R d_s$ and

$$d_s = \sqrt{s}(Z_0 + \frac{s-1}{s} D).$$

Using (3.12), the Lie brackets $[\mathfrak{m}_s, \mathfrak{m}_s]$ are determined by

$$\begin{aligned} &[d_s, e_r] = \sqrt{s}\tfrac{m+1}{\alpha_m} f_r, \quad [d_s, f_r] = -\sqrt{s}\tfrac{m+1}{\alpha_m} e_r,\\ &[e_r, e_j] = [f_r, f_j] = -B_{rj}, \quad [e_r, f_j] = C_{rj}, \quad \text{if } r \neq j,\\ &[e_r, f_r] = \tfrac{m+1}{\alpha_m}(\sqrt{s} d_s + \sqrt{(1-s)} h_s) + \tfrac{1-r}{\alpha_{r-1}} S_{r-1} + \tfrac{1}{\alpha_r} S_r + \cdots + \tfrac{1}{\alpha_{m-1}} S_{m-1}. \end{aligned} \tag{4.16}$$

Moreover, the inner product on $\mathfrak{su}(m+1) \oplus I\!R$ making

$$\{h_s, S_1, \ldots, S_{n-1}; B_{rj}, C_{rj},\ 1 \leq r < j \leq m;\ d_s, e_r, f_r,\ 1 \leq r \leq m\}$$

an orthonormal basis determines an bi-invariant metric on $SU(m+1) \times U(1)$ and makes S^{2m+1} in a $SU(m+1) \times U(1)$-normal homogeneous space isometric to (S^{2m+1}, g_s) [25]. Since $[d_s, \mathfrak{k}] = 0$, d_s is $\mathrm{Ad}(S(U(m) \times U(1)))$-invariant on $\mathfrak{m}_s$ and so, it determines a G-invariant unit vector field ξ on S^{2m+1}, the *Hopf vector field*. Because ξ is a Killing vector field, every geodesic γ_u on (S^{2m+1}, g_s) intersects each fiber with a constant slope angle $\theta = ang(\xi_o, u) \in]0, \pi[$.

For $(S^{4m+3} = Sp(m+1)/Sp(m), g_s)$, $s < 1$, let $\{D_p\}_{p=1,2,3}$ be the standard basis of $\mathfrak{sp}(1) \cong \mathfrak{su}(2)$ given by $D_1 = A_{1,2}$, $D_2 = B_{1,2}$ and $D_3 = C_{1,2}$. Put

$$d_{p_s} = \sqrt{2s}(X_p + \frac{s-1}{s} D_p), \quad h_{p_s} = \sqrt{2(1-s)}(X_p + D_p). \tag{4.17}$$

Then $\mathfrak{sp}(m+1) \oplus \mathfrak{sp}(1) = \mathfrak{k} \oplus \mathfrak{m}_s$ is a reductive decomposition associated to the quotient space $Sp(m+1) \times Sp(1)/K$, where K is the connected Lie subgroup of $Sp(m+1) \times Sp(1)$ with Lie algebra $\mathfrak{k} = \mathfrak{sp}(m) \oplus \mathfrak{sp}(1)$ generated by $\{Z_{\alpha,\beta}, Z_{\alpha p}, Z_{(\alpha,\beta)p}; h_{p_s}\}$, for $p = 1,2,3$ and $1 \leq \alpha < \beta \leq m$, and $\mathfrak{m}_s = \mathfrak{m}_{0s} \oplus \mathfrak{m}_1$ the vector space with adapted basis $\{d_{p_s};\ Y_\alpha, Y_{\alpha p}\}$. Using (3.13) and (3.14), the inner product on $\mathfrak{sp}(m+1) \oplus \mathfrak{sp}(1)$ making these vectors an orthonormal basis is bi-invariant and, in a similar way than for Berger spheres, S^{4m+3} becomes into a $Sp(m+1) \times Sp(1)$-normal homogeneous space isometric to (S^{4m+3}, g_s).

For each $\kappa > 0$, using the well-known O'Neill formula ([4, Ch. 9]) and (2.5), (3.13) and (4.16), we have the following homogeneous Riemannian fibrations:

(i) $S^1 \to (S^{2m+1}, g_{\kappa,s} = \frac{1}{\kappa} g_s) \to \mathbb{C}P^m(\kappa)$;
(ii) $S^2 \to (\mathbb{C}P^{2m+1}, g_\kappa = \frac{1}{\kappa} g) \to \mathbb{H}P^m(\kappa)$;
(iii) $S^3 \to (S^{4m+3}, g_{\kappa,s} = \frac{1}{\kappa} g_s) \to \mathbb{H}P^m(\kappa)$.

Remark 4.4. On compact symmetric spaces, the conjugate points to the origin along any geodesic γ_u are strictly isotropic and they are given by $\gamma_u(\frac{p\pi}{\sqrt{\lambda}})$, $p \in \mathbb{Z}$, where λ is a eigenvalue of R_u. In particular, on $S^n(\kappa)$ or $I\!RP^n(\kappa)$ the conjugate points to the origin are all $\gamma_u(\frac{p\pi}{\sqrt{\kappa}})$ and $\gamma_u(\frac{p\pi}{2\sqrt{\kappa}})$ on $\mathbb{K}P^n(\kappa)$, for $\mathbb{K} = \mathbb{C}$, $\mathbb{H}$ or $\mathbb{C}a$.

Now, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}_s$, $s \leq 1$, be any of the reductive decompositions previously obtained and let $\mathfrak{m}_s = \mathfrak{m}_{0s} \oplus \mathfrak{m}_1$ be the $\mathrm{Ad}(K)$-invariant orthogonal decomposition of $\mathfrak{m}_s$, where $\mathfrak{m}_{0s}$ is the L.t.s. subspace of $\mathfrak{m}_s$ associated to the fibers as totally geodesic submanifolds.

Lemma 4.5. *For $(S^{2m+1}, g_{\kappa,s})$, $(\mathbb{C}P^{2m+1}, g_\kappa)$ and $(S^{4m+3}, g_{\kappa,s})$, the linear isotropy action on $\mathcal{S} \cap \mathfrak{m}_1$ is transitive except for the Euclidean sphere $(S^3, g_{\kappa,1})$. Moreover, it is transitive on $\mathcal{S} \cap \mathfrak{m}_{0s}$ except for $(S^{4m+3}, g_{\kappa,1})$.*

Proof. Given $v = \sum_{r=1}^m (v_1^r e_r + v_2^r f_r) \in \mathfrak{m}_1 \cong \mathbb{C}^m$, (4.16) implies that

$$[e_m, v]_{\mathfrak{su}(m)} = \sum_{r=1}^{m-1} (v_1^r B_{rm} + v_2^r C_{rm}) - v_2^m \frac{m-1}{\alpha_{m-1}} S_{m-1}$$

and, putting $v = \sum^m_{\substack{\alpha=1 \\ p=1,2,3}} (v^\alpha Y_\alpha + v^{\alpha p} Y_{\alpha p}) \in \mathfrak{m}_1 \cong \mathbb{H}^m$, it follows from (3.13), (3.14)

$$[Y_1, v]_{\mathfrak{sp}(m)} = \sum_{\alpha=2}^m (-v^\alpha Z_{1,\alpha} + \sum_{p=1,2,3} v^{\alpha p} Z_{(1,\alpha)p}) + 2 \sum_{p=1,2,3} v^{1p} Z_{1p}.$$

Then $[e_m, v]_{\mathfrak{su}(m)}$, for $m > 1$, or $[Y_1, v]_{\mathfrak{sp}(m)}$ is zero if and only if u is collinear to e_m or v to Y_1. For $(S^3, g_{\kappa,s})$, we obtain $[e_1, v]_{\mathfrak{k}} = 2\sqrt{1-s}\, v_2^1 h_s$. Hence, using Proposition 4.2, the linear isotropy action on $\mathcal{S} \cap \mathfrak{m}_1$ is transitive in each case. Also the linear isotropy action on $\mathcal{S} \cap \mathfrak{m}_{0s}$ is transitive for $(S^{2m+1}, g_{\kappa,s})$ and $(\mathbb{C}P^{2m+1}, g_\kappa)$. On $(S^{4m+3}, g_{\kappa,s})$, taking $x_s = \sum_{p=1}^3 x_s^p d_{p_s}$ one gets

$$[d_{1s}, x_s]_{\mathfrak{k}} = 2\sqrt{2(1-s)}(x_s^2 h_{3s} - x_s^3 h_{2s}).$$

Hence, using again Proposition 4.2, the last part of the lemma is proved. □

Using (2.5) and (3.13) and (4.16), it follows

Lemma 4.6. *The sectional curvature $K(u_0, u_1)$, for all $u_0 \in \mathfrak{m}_{0s}$ and $u_1 \in \mathfrak{m}_1$, is a function $\tau = \tau(\kappa, s) = \frac{\kappa s(m+1)}{2m}$ on $(S^{2m+1}, g_{\kappa,s})$, $\tau = \frac{\kappa}{2}$ on $(\mathbb{C}P^{2m+1}, g_\kappa)$ and $\tau = \frac{\kappa s}{2}$ on $(S^{4m+3}, g_{\kappa,s})$.*

Remark 4.7. $\tau < \kappa$ except in the 3-dimensional Euclidean sphere $(S^3, g_{\kappa,1})$, where $\tau = \kappa$.

Next, we shall prove the following result, which generalizes one given in [8] for Berger spheres.

Theorem 4.8. *On $(S^{2m+1}, g_{\kappa,s})$ and $(S^{4m+3}, g_{\kappa,s})$, for $s \leq 1$, and on $(\mathbb{C}P^{2m+1}, g_\kappa)$ any geodesic γ_u, $u = u(\theta)$, admits conjugate points which are not strictly isotropic and there exist geodesics admitting non-isotropic conjugate points. Concretely, we have:*

(i) *If γ_u is a vertical geodesic, i.e. $\theta = 0$, the points $\gamma_u(\frac{p\pi}{\sqrt{\tau}})$, $p \in \mathbb{Z}$, are conjugate points but not strictly isotropic. On $(S^{2m+1}, g_{\kappa,s})$ these are all conjugate points along the Hopf fiber γ_u and they are not isotropic.*

(ii) *If $\theta \in]0, \frac{\pi}{2}]$, the points of the form $\gamma_u(\frac{t}{2\sqrt{\kappa \sin^2\theta + \tau\cos^2\theta}})$, where*

   (A) *t is a solution of the equation $\tan\frac{t}{2} = \frac{t(\tau-\kappa)\sin^2\theta}{2\tau}$, or*

   (B) *$t = 2p\pi$, $p \in \mathbb{Z}$,*

   are conjugate points to the origin. In the first case, they are not strictly isotropic and in the second one, they are isotropic.

(iii) *If γ_u is a horizontal geodesic, $\theta = \frac{\pi}{2}$, the points $\gamma_u(\frac{t}{2\sqrt{\kappa}})$ as in* (ii) (A)*, where t is a solution of the equation $\tan\frac{t}{2} = \frac{t(\tau-\kappa)}{2\tau}$, are conjugate points which are not isotropic.*

Proof. We shall show that in each one of these spaces there exist orthonormal vectors $u, v \in \mathfrak{m}_s$ satisfying the conditions of Theorem 2.9 (ii) for coefficients λ and ρ given by $\lambda = 4\tau$ and $\rho = 4(\kappa-\tau)\sin^2\theta$:

On Berger spheres $(S^{2m+1}, g_{\kappa,s})$, by using of Lemma 4.5, we restrict our study to geodesics γ_u starting at the origin with $u = \sqrt{\kappa}(\cos\theta d_s + \sin\theta e_m)$. Put $v = \sqrt{\kappa}(\cos\theta e_m - \sin\theta d_s)$. Then, using (4.16) and Lemma 4.6, we get

$$[u,v] = \kappa\sqrt{s}\tfrac{m+1}{\alpha_m} f_m, \quad [[u,v],u]_{\mathfrak{m}_s} = 2\kappa s\tfrac{m+1}{m} v = 4\tau v,$$
$$[u,[u,v]]_{\mathfrak{k}} = \kappa\sqrt{\kappa s}\tfrac{m+1}{\alpha_m}\sin\theta\Big(\tfrac{m+1}{\alpha_m}\sqrt{1-s}h_s + \tfrac{1-m}{\alpha_{m-1}}S_{m-1}\Big).$$

Moreover, from (2.2) and (4.16), $[[u,[u,v]]_{\mathfrak{k}}, u]$ is collinear to f_m. It proves the result for this case. On $(\mathbb{C}P^{2m+1}, g_\kappa)$, we only need to consider geodesics γ_u with u given by $u = \sqrt{\kappa}(\cos\theta X + \sin\theta Y_\alpha)$, for some $\alpha \in \{1,\dots,m\}$, where $X = X(\phi) = \sqrt{2}(\cos\phi X_2 + \sin\phi X_3) \in \mathfrak{p}_0 = \mathfrak{m}_0$. Put $v = \sqrt{\kappa}(\cos\theta Y_\alpha - \sin\theta X)$ in Theorem 2.9. Then, using (3.13) and (3.14), we get

$$\begin{aligned} [u,v] &= -\sqrt{2}\kappa(\cos\phi Y_{\alpha 2} + \sin\phi Y_{\alpha 3}), \\ [[u,v],u] &= 2\kappa(v + \sqrt{2\kappa}\sin\theta(\cos\phi Z_{\alpha 2} + \sin\phi Z_{\alpha 3})) \end{aligned}$$

and $[[[u,v],u]_{\mathfrak{k}}, u]$ is collinear to $[u,v]$. Finally, on $(S^{4m+3}, g_{\kappa,s})$, using again Lemma 4.5, we can take geodesics γ_u with $u = \sqrt{\kappa}(\cos\theta X_s + \sin\theta Y_\alpha)$, for some $\alpha \in \{1,\dots,m\}$, where X_s is an arbitrary vector of $\mathfrak{m}_{0s}$, written as $X_s = X_s(\phi_1,\phi_2) = \sin\phi_1\cos\phi_2 d_{1s} + \sin\phi_1\sin\phi_2 d_{2s} + \cos\phi_2 d_{3s}$. Now, put $v = \sqrt{\kappa}(\cos\theta Y_\alpha - \sin\theta X_s)$. Then, from (3.13), (3.14) and (4.17), taking into account Lemma 4.6, one gets

$$\begin{aligned} [u,v] &= -\sqrt{2s}\kappa(\sin\phi_1\cos\phi_2 Y_{\alpha 1} + \sin\phi_1\sin\phi_2 Y_{\alpha 2} + \cos\phi Y_{\alpha 3}), \\ [[u,v],u]_{\mathfrak{m}_s} &= 2\kappa s v = 4\tau v, \\ [u,[u,v]]_{\mathfrak{k}} &= 2\kappa\sqrt{\kappa s}\sin\theta\Big((\sqrt{1-s}h_{1s} - \sqrt{2}Z_{\alpha 1})\sin\phi_1\cos\phi_2 \\ &\quad + (\sqrt{1-s}h_{2s} - \sqrt{2}Z_{\alpha 2})\sin\phi_1\sin\phi_2 + (\sqrt{1-s}h_{3s} - \sqrt{2}Z_{\alpha 3})\cos\phi_2\Big), \\ [[u,[u,v]]_{\mathfrak{k}}, u] &= 2(2-s)\kappa\sin^2\theta[u,v]. \end{aligned}$$

Hence, the result also holds for this last case.

If γ_u is vertical, $\rho = 0$ and (i) follows from Theorem 2.9 (ii)(A). On $(S^{2m+1}, g_{\kappa,s})$, the vector $d_s \in \mathfrak{m}_s$ determines the Hopf vector field, which is $(S\,U(m+1)\times U(1))$-invariant. From (2.3), (2.4) and (4.16), one gets that $\{e_r, f_r\}$ generates the eigenspace of the Jacobi operator R_{d_s} with eigenvector $\frac{\tau}{\kappa}$. Then, using [12, Theorem 5.3], the points $\gamma_u(\frac{p\pi}{\sqrt{\tau}})$ are all conjugate points along the Hopf fibers, their multiplicity is $2m$ and they are not isotropically conjugate. For (ii) we use directly Theorem 2.9 (ii) (B). Finally, (iii) follows from (ii) and Lemma 4.1 and Remark 4.4. □

Corollary 4.9. *Any vertical geodesic in $(\mathbb{C}P^{2m+1}, g_\kappa)$ admits isotropic conjugate points which are not strictly isotropic.*

Proof. Put $u = \sqrt{2\kappa}(\cos\phi X_2 + \sin\phi X_3)$ and $v = \sqrt{2\kappa}(\cos\phi X_3 - \sin\phi X_2)$ in Theorem 2.9. Then, from (3.13), one gets $[u, v] = 4\kappa X_1$ and $[[u, v], v] = 8\kappa v$. So, the points $\gamma_u(\frac{\sqrt{2\kappa}p\pi}{4\kappa})$ are isotropically conjugate to the origin. Hence, using Theorem 4.8 (i), the points $\gamma_u(\frac{\sqrt{2\kappa}p\pi}{\kappa})$ satisfy the conditions of this corollary. □

Remark 4.10. From (2.5) and (3.13), the 2-sphere $S^2 = Sp(1)/U(1)$, isometrically embedded as totally geodesic submanifold in $(\mathbb{C}P^{2m+1}, g_\kappa)$, is the Euclidean sphere with constant curvature 8κ. Then the vertical geodesics of $(\mathbb{C}P^{2m+1}, g_\kappa)$ are closed with length $\frac{\sqrt{2\kappa}\pi}{2\kappa}$.

4.2. The Berger space $B^{13} = SU(5)/H$.

The isotropy subgroup H is given by

$$H := \left\{ \begin{pmatrix} zA & 0 \\ 0 & \bar{z}^4 \end{pmatrix} \mid A \in Sp(2) \subset SU(4),\ z \in S^1 \subset \mathbb{C} \right\} \subset S(U(4) \times U(1)) \subset SU(5).$$

Then H may be considered as the image of $Sp(2) \times S^1$ under the homomorphism

$$\chi : SU(4) \times S^1 \to U(4) \subset SU(5), \quad (A, z) \mapsto zA,$$

where $U(4)$ is embedded in $SU(5)$ through the map $\iota : A \to \begin{pmatrix} A & 0 \\ 0 & \det A^{-1} \end{pmatrix}$ or equivalently, as $(Sp(2) \times S^1)/\{\pm(Id, 1)\}$. So, its Lie algebra $\mathfrak{h}$ is given by $\mathfrak{h} = \mathfrak{sp}(2) \oplus I\!RZ_0$, where $Z_0 = S_4$. On $\mathfrak{su}(5)$ we take the bi-invariant inner product $\langle X, Y\rangle = -\frac{1}{4}\text{trace}\, XY$. Using the fact that $Sp(2)/\mathbb{Z}_2 \cong SO(5)$ and $SU(4)/\mathbb{Z}_2 \cong SO(6)$, it follows that the five-dimensional Euclidean sphere $S^5 = SO(6)/SO(5)$ can be also described as the quotient $SU(4)/Sp(2)$, with reductive orthogonal decomposition $\mathfrak{su}(4) = \mathfrak{sp}(2) \oplus \mathfrak{m}_0$, $\mathfrak{m}_0 \cong I\!R^5$. Moreover, because $[Z_0, \mathfrak{m}_0] = 0$, one gets that $[\mathfrak{h}, \mathfrak{m}_0] = [\mathfrak{sp}(2), \mathfrak{m}_0] \subset \mathfrak{m}_0$.

On the other hand, the quotient expression $SU(5)/S(U(4) \times U(1))$ for $\mathbb{C}P^4$ determines another reductive decomposition $\mathfrak{su}(5) = (\mathfrak{su}(4) \oplus I\!RZ_0) \oplus \mathfrak{m}_1$, $\mathfrak{m}_1 \cong \mathbb{C}^4$. Then, $[\mathfrak{h}, \mathfrak{m}_1] \subset [\mathfrak{su}(4) \oplus I\!RZ_0, \mathfrak{m}_1] \subset \mathfrak{m}_1$. Hence, $\mathfrak{su}(5) = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1$ is a reductive decomposition of $\mathfrak{su}(5)$ associated to $SU(5)/H$. Since H is connected, $\mathfrak{m}_0$ and $\mathfrak{m}_1$ are $Ad(H)$-invariant subspaces of $\mathfrak{m}$ and, because $\mathfrak{h}$ contains to $\mathfrak{sp}(2)$, $\mathfrak{m}_0$ is isotropy-irreducible. Moreover, the corresponding totally geodesic submanifold $M_{\mathfrak{m}_0}$ through the origin in B^{13} is given by $SU(4)/(SU(4) \cap H)$, which is diffeomorphic to the 5-dimensional real projective space $I\!RP^5$ because $\chi(A, 1) = \chi(-A, -1)$, for all $A \in Sp(2)$.

Proposition 4.11. *The projection* $\pi : B^{13} \to \mathbb{C}P^4 = SU(5)/S(U(4) \times U(1))$, $gH \mapsto gS(U(4) \times U(1))$, *determines a homogeneous fibration* $I\!RP^5 \to B^{13} \to \mathbb{C}P^4$, *where the fibers are obtained by the action of* $SU(5)$ *on* $I\!RP^5 = SU(4)/(SU(4) \cap H)$.

Proof. Because $S(U(4) \times U(1))$ is closed in $SU(5)$ and H compact, it follows that the quotient manifold $S(U(4) \times U(1))/H$ is a regular submanifold of B^{13} (see [15, Proposition 4.4, Ch. II]) and $\mathfrak{su}(4) \oplus \mathfrak{u}(1) = (\mathfrak{sp}(2) \oplus I\!RZ_0) \oplus \mathfrak{m}_1$ is a reductive decomposition. Moreover, $SU(4)/(SU(4) \cap H)$ is a submanifold of $S(U(4) \times U(1))/H$, via the immersion $g\ (SU(4) \cap H) \mapsto \iota(g)H$, for $g \in SU(4)$, where $\iota : SU(4) \to S(U(4) \times U(1))$ is the standard inclusion map. Hence, $SU(4)/(SU(4) \cap H)$ becomes into an open submanifold of $S(U(4) \times U(1))/H$. Taking into account that it is also closed in $S(U(4) \times U(1))/H$, one

gets

$$\mathbb{R}P^5 = \frac{SU(4)}{SU(4)\cap H} = \frac{S(U(4)\times U(1))}{H}.$$

It proves the proposition. $\square$

Following [14], an orthonormal basis of $\mathfrak{su}(5)$ with respect to $\langle\cdot,\cdot\rangle$ and adapted to the reductive decomposition $\mathfrak{su}(5) = \mathfrak{h}\oplus\mathfrak{m}_0\oplus\mathfrak{m}_1$ is

$$\{H_1,\dots,H_{11}; u_0,u_1,u_2,v_1,v_2; e_1,\dots,e_4,f_1,\dots,f_4\}$$

where $\{H_1,\dots,H_{11}\}$ is the basis of $\mathfrak{h}$ given by

$$\begin{array}{lllllllll}
H_1 &=& A_{1,2}+2A_{2,3}+A_{3,4}, & H_2 &=& B_{1,3}+B_{2,4}, & H_3 &=& C_{1,3}+C_{2,4},\\
H_4 &=& A_{1,2}-A_{3,4}, & H_5 &=& B_{1,3}-B_{2,4}, & H_6 &=& C_{1,3}-C_{2,4},\\
H_7 &=& C_{1,2}-C_{3,4}, & H_8 &=& B_{1,4}+B_{2,3}, & H_9 &=& C_{1,4}+C_{2,3},\\
H_{10} &=& B_{1,2}+B_{3,4}, & H_{11} &=& \sqrt{2}Z_0, & & &
\end{array}$$

the matrices A_{jk}, B_{jk}, C_{jk}, $1\le j<k\le 5$, are defined in (3.11),

$$u_0 = A_{1,2}+A_{3,4},\quad u_1 = B_{1,2}-B_{3,4},\quad u_2 = B_{1,4}-B_{2,3},\quad v_1 = C_{1,2}+C_{3,4},\quad v_2 = C_{1,4}-C_{2,3}$$

constitute an orthonormal basis for $\mathfrak{m}_0$ and

$$e_r = \sqrt{2}B_{r,5},\quad f_r = J_o e_r = \sqrt{2}C_{r,5},\quad r=1,\dots,4,$$

is a basis of $\mathfrak{m}_1 \cong \mathbb{C}^4$, being J_o the canonical complex structure $J_o = \frac{1}{\sqrt{5}}\mathrm{ad}_{Z_0}$ induced in $\mathbb{C}P^4$.

Every $SU(5)$-invariant metric on B^{13} is determined, up to a scaling factor, by an $Ad(H)$-invariant inner product $\langle\cdot,\cdot\rangle_s$ on $\mathfrak{m}$, for some $s>0$, given by [19]

$$\langle\cdot,\cdot\rangle_s = s\langle\cdot,\cdot\rangle_{\mathfrak{m}_0\times\mathfrak{m}_0} + \langle\cdot,\cdot\rangle_{\mathfrak{m}_1\times\mathfrak{m}_1}.$$

Lemma 4.12. *The standard metric $\langle\cdot,\cdot\rangle$ is, up to homotheties, the unique $SU(5)$-invariant metric on B^{13} which is normal homogeneous.*

Proof. We shall show that s must be 1. Suppose that $\langle\cdot,\cdot\rangle_s$ can be extended to a bi-invariant inner product of $\mathfrak{su}(5)$, which we also denote by $\langle\cdot,\cdot\rangle_s$, making $\mathfrak{h}$ and $\mathfrak{m}$ orthogonal. Then, from (2.2) and (3.12), we have

$$1 = \langle e_1,e_1\rangle_s = \langle [u_2,e_4],e_1\rangle_s = \langle [e_4,e_1],u_2\rangle_s = \langle u_2+H_8,u_2\rangle_s = \langle u_2,u_2\rangle_s = s.$$

$\square$

Using (3.12), one gets $\|[e_r,f_r]_{\mathfrak{h}}\|^2 = 7$ and $\|[e_r,f_r]_{\mathfrak{m}}\|^2 = \|[e_r,f_r]_{\mathfrak{m}_0}\|^2 = 1$, for $r=1,\dots,4$. Then, from (2.5) and the O'Neill formula, it follows

$$K_{B^{13}}(e_r,f_r) = \frac{29}{4},\quad K_{\mathbb{C}P^4}(e_r,f_r) = 8.$$

Hence, the base space of the Riemannian submersion in Proposition 4.11 is isometric to $\mathbb{C}P^4(2)$. Moreover, since for orthonormal vectors $u,v\in\mathfrak{m}_0$, one gets $\|[u,v]\|^2 = \|[u,v]_{\mathfrak{h}}\|^2 = 4$, (2.5) implies that the fibers are isometric to $\mathbb{R}P^5(4)$.

Lemma 4.13. *The linear isotropy action of H on $\mathcal{S}\cap\mathfrak{m}_0$ and on $\mathcal{S}\cap\mathfrak{m}_1$ is transitive.*

Proof. Given $x = x^0u_0 + \sum_{i=1,2}(x_1^iu_i + x_2^iv_i) \in \mathfrak{m}_0$ and $v = \sum_{r=1}^4(v_1^re_r + v_2^rf_r) \in \mathfrak{m}_1$, one gets, using (3.12), the following brackets:

$$\begin{aligned}[][u_0,x]_{\mathfrak{h}} &= 2(x_1^1H_7 - x_2^2H_8 + x_1^2H_9 - x_2^1H_{10}),\\ [e_1,v]_{\mathfrak{h}} &= v_2^1H_1 - v_1^3H_2 + v_2^3H_3 + v_2^1H_4 - v_1^3H_5 + v_2^3H_6 + v_2^2H_7\\ &\quad -v_1^4H_8 + v_2^4H_9 - v_1^2H_{10} + \sqrt{5}v_2^1H_{11}.\end{aligned}$$

It implies that $[u_0,x]_{\mathfrak{h}}$ or $[e_1,v]_{\mathfrak{h}}$ is zero if and only if x is collinear to u_0 or v is collinear to e_1. Then the result follows from Proposition 4.2. □

Theorem 4.14. *Let γ_u, $u = u(\theta)$, be a geodesic on (B^{13}, g) with slope angle $\theta = ang(u,\mathfrak{m}_0) \in [0,\frac{\pi}{2}]$. Then we have:*

(i) *If γ_u is a vertical geodesic, the points $\gamma_u(\frac{p\pi}{2})$, $p \in \mathbb{Z}$, are isotropically conjugate to the origin and those $\gamma_u(p\pi)$ are not strictly isotropic.*

(ii) *If $\theta \in]0,\frac{\pi}{2}]$ and u is orthogonal to u_0, $\gamma_u(\frac{t}{\sqrt{1+\sin^2\theta}})$, where*

(A) *t is a solution of the equation $\tan\frac{t}{2} = -\sin^2\theta\frac{t}{2}$ or*

(B) *$t = 2p\pi$, $p \in \mathbb{Z}$,*

are conjugate points to the origin. In the first case, they are not strictly isotropic and in the second one, they are isotropic.

(iii) *If γ_u is a horizontal geodesic, the points $\gamma_u(\frac{\sqrt{2}t}{2})$ as in* (ii) A, *where t is a solution of the equation $\tan\frac{t}{2} = -\frac{t}{2}$, are conjugate points to the origin but not isotropic.*

Proof. From above lemma, the study of conjugate points along γ_u can be reduced to consider u as $u = u(\theta) = \cos\theta x + \sin\theta e_1$, where $x = x^0u_0 + \sum_{i=1,2}(x_1^iu_i + x_2^iv_i) \in \mathfrak{m}_0$ and $x_0^2 + \sum_{i,j=1,2}(x_j^i)^2 = 1$. Put $v = \cos\theta e_1 - \sin\theta x$. Then, from (3.12), one gets

$$[u,v] = [x,e_1] = -x_1^1e_2 - x_1^2e_4 + x^0f_1 + x_2^1f_2 + x_2^2f_4$$

and

$$\begin{aligned}[][[u,v],u] &= x^0(\cos\theta(x^0e_1 + x_2^1e_2 + x_2^2e_4 + x_1^1f_2 + x_1^2f_4) - \sin\theta(u_0 + H_1 + H_4 + \sqrt{5}H_{11}))\\ &\quad +x_1^1(\cos\theta(x_1^1e_1 + x_1^2e_3 + x_2^1f_1 - x^0f_2 - x_2^2f_3) - \sin\theta(u_1 + H_{10}))\\ &\quad +x_1^2(\cos\theta(x_1^2e_1 - x_1^1e_3 + x_2^2f_1 + x_2^1f_3 - x^0f_4) - \sin\theta(u_2 + H_8))\\ &\quad +x_2^1(\cos\theta(x_2^1e_1 - x^0e_2 - x_2^2e_3 - x_1^1f_1 - x_1^2f_3) - \sin\theta(v_1 + H_7))\\ &\quad +x_2^2(\cos\theta(x_2^2e_1 + x_2^1e_3 - x^0e_4 - x_1^2f_1 + x_1^1f_3) - \sin\theta(v_2 + H_9)).\end{aligned}$$

Hence, it follows

$$[[u,v],u]_{\mathfrak{m}} = v, \quad [u,[u,v]]_{\mathfrak{k}} = \sin\theta(x^0(H_1 + H_4 + \sqrt{5}H_{11}) + x_1^1H_{10} + x_1^2H_8 + x_2^1H_7 + x_2^2H_9).$$

Moreover, one gets

$$[[u,[u,v]]_{\mathfrak{k}},u] = \sin^2\theta([u,v] + 6x^0f_1). \tag{4.18}$$

If γ_u is a vertical geodesic, $\rho = 0$ and u,v satisfy the hypothesis of Theorem 2.9 for $\lambda = 1$. So, $\gamma_u(p\pi)$, $p \in \mathbb{Z}$, are not strictly isotropic conjugate points. Using again Lemma 4.13, we can take the geodesic γ_u, with $u = u_0$. From (3.12) we get

$$[u_0,u_1] = 2H_7, \quad [[u_0,u_1],u_0] = 4u_1.$$

Then Theorem 2.9 (i) yields that $\gamma_u(\frac{p\pi}{2})$, $p \in \mathbb{Z}$, are isotropically conjugate points to the origin. It proves (i).

For $\theta \in]0, \pi/2]$, if $x^0 = 0$, or equivalently u is orthogonal to u_0, (4.18) implies that $[[u,[u,v]]_{\mathfrak{k}}, u]$ is collinear to $[u,v]$. Then (ii) follows using Theorem 2.9 (ii) (B). Finally, using Lemma 4.1 and taking into account Remark 4.4 and that the base space of the canonical homogeneous Riemannian fibration is isometric to $\mathbb{C}P^4(2)$, the proof of (iii) is completed. □

4.3. The Wilking's example $W^7 = (SO(3) \times SU(3))/U^\bullet(2)$.

$U^\bullet(2)$ denotes the image of $U(2)$ under the embedding $(\pi, \iota) : U(2) \hookrightarrow SO(3) \times SU(3)$, where π is the projection $\pi : U(2) \to U(2)/S^1 \cong SO(3)$, being $S^1 \subset U(2)$ the center of $U(2)$, and $\iota : U(2) \to SU(3)$ the natural inclusion (see Section 4.2). Using the natural isomorphism between $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$, we can consider the Lie algebra $\mathfrak{so}(3) \oplus \mathfrak{su}(3)$ of $SO(3) \times SU(3)$ as the subalgebra of $\mathfrak{su}(5)$ of matrices of the form

$$X = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \qquad X_1 \in \mathfrak{su}(2),\ X_2 \in \mathfrak{su}(3).$$

Then the Lie algebra $\mathfrak{u}^\bullet(2)$ of $U^\bullet(2)$ is given by

$$\mathfrak{u}^\bullet(2) = \left\{ (\pi_* A, A) = \left(\begin{array}{c|cc} \pi_* A & 0 & 0 \\ \hline 0 & A & 0 \\ 0 & 0 & -\operatorname{trace} A \end{array} \right) \mid \quad A \in \mathfrak{u}(2) \right\},$$

where π_* denotes the differential map of π. It implies that $\mathfrak{u}^\bullet(2) \cong \Delta(\mathfrak{su}(2)) \oplus I\!RZ_0$, where $\Delta(\mathfrak{su}(2))$ is the Lie subalgebra in $\mathfrak{u}^\bullet(2)$ defined by $\Delta(\mathfrak{su}(2)) = \{(X,X) \mid X \in \mathfrak{su}(2)\}$ and $Z_0 \cong (0, Z_0)$ is a generator of the centralizer of $\mathfrak{su}(2)$ in $\mathfrak{su}(3) \cong (0, \mathfrak{su}(3))$. On $\mathfrak{su}(5)$ we take the bi-invariant inner product given by $\langle X, Y\rangle = -\frac{1}{2}\operatorname{trace} XY$. $SO(3)$ as symmetric space is isometric to $I\!RP^3$ and it can be expressed as the quotient $SO(3) \times SO(3)/\Delta(SO(3))$, being $\Delta(SO(3))$ the diagonal of $SO(3) \times SO(3)$. Then we get the corresponding reductive orthogonal decomposition $\mathfrak{su}(2) \oplus \mathfrak{su}(2) = \Delta(\mathfrak{su}(2)) \oplus \mathfrak{m}_0$, where $\mathfrak{m}_0 = \{(X, -X) \mid X \in \mathfrak{su}(2)\}$.

On the other hand, the quotient expression $SU(3)/S(U(2) \times U(1))$ for $\mathbb{C}P^2$ determines another reductive decomposition $\mathfrak{su}(3) = (\mathfrak{su}(2) \oplus I\!RZ_0) \oplus \mathfrak{m}_1$, $\mathfrak{m}_1 \cong \mathbb{C}^2$. Then, identifying $\mathfrak{m}_1$ with $(0, \mathfrak{m}_1)$, we have $[\mathfrak{u}^\bullet(2), \mathfrak{m}_1] \subset (0, [\mathfrak{su}(2) \oplus I\!RZ_0, \mathfrak{m}_1]) \subset \mathfrak{m}_1$. Hence, one gets

$$\begin{aligned} \mathfrak{so}(3) \oplus \mathfrak{su}(3) &\cong \mathfrak{su}(2) \oplus \mathfrak{su}(3) = (\mathfrak{su}(2) \oplus \mathfrak{su}(2)) \oplus I\!RZ_0 \oplus \mathfrak{m}_1 \\ &= (\Delta(\mathfrak{su}(2)) \oplus I\!RZ_0) \oplus \mathfrak{m}_0 \oplus \mathfrak{m}_1 \cong \mathfrak{u}^\bullet(2) \oplus \mathfrak{m}_0 \oplus \mathfrak{m}_1. \end{aligned}$$

Then $\mathfrak{so}(3) \oplus \mathfrak{su}(3) = \mathfrak{u}^\bullet(2) \oplus \mathfrak{m}$, where $\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1$, is a reductive decomposition associated to $(SO(3) \times SU(3))/U^\bullet(2)$. Since $U^\bullet(2)$ is connected, it follows that $\mathfrak{m}_0$ and $\mathfrak{m}_1$ are $Ad(U^\bullet(2))$-invariant subspaces of $\mathfrak{m}$.

Any left-invariant metric on $SO(3) \times SU(3)$ is bi-invariant and, up to a scaling factor, they are given by the one-parameter family of inner products $\langle \cdot, \cdot \rangle_s := s\langle \cdot, \cdot \rangle_{\mathfrak{so}(3)\times\mathfrak{so}(3)} + \langle \cdot, \cdot \rangle_{\mathfrak{su}(3)\times\mathfrak{su}(3)}$, for some $s > 0$, on $\mathfrak{so}(3) \oplus \mathfrak{su}(3)$. The induced metrics g_s on W^7 which turn the projection $SO(3) \times SU(3) \to W^7$ into a Riemannian submersion are normal homogeneous and they are determined by the restriction of $\langle \cdot, \cdot \rangle_s$ to the orthogonal complement $\mathfrak{m}_s$ of $\mathfrak{u}^\bullet(2)$ on $(\mathfrak{so}(3) \oplus \mathfrak{su}(3), \langle \cdot, \cdot \rangle)$. It can be expressed as $\mathfrak{m}_s = \mathfrak{m}_{0s} \oplus \mathfrak{m}_1$, where $\mathfrak{m}_{0s} = \{(X, -sX) \mid X \in \mathfrak{su}(2)\}$. Here, $\mathfrak{m}_{0s}$ is a Lie triple system with associated totally geodesic diffeomorphic to $I\!RP^3$.

An orthonormal basis $\{K_1, K_2, K_3, K_4; u_{0s}, u_{1s}, v_{1s}; e_1, e_2, f_1, f_2\}$ of $(\mathfrak{so}(3) \oplus \mathfrak{su}(3), \langle\cdot,\cdot\rangle_s)$ adapted to the reductive decomposition $\mathfrak{u}^\bullet(2) \oplus (\mathfrak{m}_{0s} \oplus \mathfrak{m}_1)$ is given as follows:

$$\begin{array}{rclrcl} K_1 & = & \frac{1}{\sqrt{1+s}}(A_{1,2} + A_{3,4}), & K_2 & = & \frac{1}{\sqrt{1+s}}(B_{1,2} + B_{3,4}), \\ K_3 & = & \frac{1}{\sqrt{1+s}}(C_{1,2} + C_{3,4}), & K_4 & = & \frac{1}{\sqrt{3}}(A_{3,4} + 2A_{4,5}), \\ u_{0s} & = & \frac{1}{\sqrt{s(1+s)}}(A_{1,2} - sA_{3,4}), & u_{1s} & = & \frac{1}{\sqrt{s(1+s)}}(B_{1,2} - sB_{3,4}), \\ v_{1s} & = & \frac{1}{\sqrt{s(1+s)}}(C_{1,2} - sC_{3,4}), & e_i & = & B_{i+2,5}, \\ f_i & = & C_{i+2,5}, \quad i = 1,2. & & & \end{array}$$

(See [11] for the brackets $[\mathfrak{m},\mathfrak{m}]$.) Each Riemannian space (W^7, g_s) is isometric to the Aloff-Wallach space $(M^7_{11}, \tilde{g}_t)$, for $t = -\frac{3}{2s+3}$ [24], and, in similar way than in above section, we have a homogeneous Riemannian fibration $\mathbb{R}P^3 \to W^7 \to \mathbb{C}P^2$. Moreover, since for orthonormal vectors $u, v \in \mathfrak{m}_{0s}$ one gets

$$\|[u,v]_{\mathfrak{m}_s}\|^2 = \frac{4(1-s)^2}{s(1+s)}, \quad \|[u,v]_{\mathfrak{u}^\bullet(2)}\|^2 = \frac{4}{1+s},$$

formula (2.5) implies that the fibers as totally geodesic submanifold are isometric to $\mathbb{R}P^3(\frac{1+s}{s})$.

Lemma 4.15. *The linear isotropy action of $U^\bullet(2)$ on $\mathcal{S} \cap \mathfrak{m}_{0s}$ and on $\mathcal{S} \cap \mathfrak{m}_1$ is transitive.*

Proof. Given $x = x^0 u_{0s} + x^1_1 u_{1s} + x^1_2 v_{1s} \in \mathfrak{m}_{0s}$ and $v = \sum_{r=1}^2 (v^r_1 e_r + v^r_2 f_r) \in \mathfrak{m}_1$, one gets, using (3.12), the following brackets:

$$\begin{array}{rcl} [u_{0s}, x]_{\mathfrak{u}^\bullet(2)} & = & \frac{2}{\sqrt{1-s}}(-x^1_2 K_2 + x^1_1 K_3), \\ [e_1, v]_{\mathfrak{u}^\bullet(2)} & = & \frac{1}{\sqrt{1+s}}(v^1_2 K_1 - v^2_1 K_2 + v^2_2 K_3 + \sqrt{3(1+s)}v^1_2 K_4). \end{array}$$

It implies that $[u_{0s}, x]_{\mathfrak{u}^\bullet(2)}$ or $[e_1, v]_{\mathfrak{u}^\bullet(2)}$ is zero if and only if x is collinear to u_{0s} or v is collinear to e_1. Then the result follows from Proposition 4.2. □

Theorem 4.16. *Let γ_u, $u = u(\theta)$, be a geodesic on (W^7, g_s) with slope angle $\theta = ang(u, \mathfrak{m}_{0s}) \in [0, \frac{\pi}{2}]$. Then we have:*

(i) *If γ_u is a vertical geodesic, the points $\gamma_u(2\sqrt{\frac{s}{1+s}}p\pi)$, $p \in \mathbb{Z}$, are isotropic conjugate to the origin and those $\gamma_u(2\sqrt{\frac{1+s}{s}}p\pi)$, are not strictly isotropic;*

(ii) *If $\theta \in]0, \frac{\pi}{2}]$ and u is orthogonal to u_{0s}, the points of the form $\gamma_u(\frac{t\sqrt{1+s}}{\sqrt{s+\sin^2\theta}})$, where*
   (A) *t is a solution of the equation $\tan\frac{t}{2} = -\sin^2\theta\frac{t}{2s}$, or*
   (B) *$t = 2p\pi$, $p \in \mathbb{Z}$,*
   are conjugate to the origin. In the first case, they are not strictly isotropic and in the second one, they are isotropic.

(iii) *If γ_u is a horizontal geodesic, the points $\gamma_u(t)$ in* (ii) A*, where t is a solution of the equation $\tan\frac{t}{2} = -\frac{t}{2s}$, are conjugate to the origin but not isotropic.*

Proof. From the transitivity of the isotropy action on $\mathcal{S}\cap\mathfrak{m}_1$ proved in Lemma 4.15, the study of conjugate points along γ_u can be reduced to consider u as $u = u(\theta) = \cos\theta x + \sin\theta e_1$,

where $x = x^0 u_{0s} + x_1^1 u_{1s} + x_2^1 v_{1s} \in \mathfrak{m}_{0s}$, $(x^0)^2 + (x_1^1)^2 + (x_2^1)^2 = 1$. Put $v = \cos\theta e_1 - \sin\theta x$. From (3.12), one gets

$$[u, v] = [x, e_1] = \sqrt{\frac{s}{1+s}}(x_1^1 e_2 - x^0 f_1 - x_2^1 f_2)$$

and

$$[[u, v], u] = \frac{s}{1+s}(\cos\theta e_1 - \sin\theta x) + \frac{\sqrt{s}}{1+s}\sin\theta(x^0(K_1 + \sqrt{3(1+s)}K_4) + x_1^1 K_2 + x_2^1 K_3).$$

Hence, $[[u, v], u]_{\mathfrak{m}_s} = \frac{s}{1+s}v$ and u and v satisfy the conditions of Theorem 2.9 for $\lambda = \frac{s}{1+s}$ and $\rho = \frac{1}{1+s}\sin^2\theta(1 + 3(1+s)(x^0)^2)$. Moreover, one gets

$$[[u, [u, v]]_{\mathfrak{u}^\bullet(2)}, u] = \frac{\sin^2\theta}{1+s}([u, v] - 3\sqrt{s(1+s)}x^0 f_1). \tag{4.19}$$

If γ_u is vertical then $\rho = 0$ and so the points $\gamma_u(2\sqrt{\frac{1+s}{s}}p\pi)$, $p \in \mathbb{Z}$, are non-strictly isotropic conjugate points to the origin. Since in $I\!RP^3(\frac{1+s}{s})$, $\gamma_u(2\sqrt{\frac{s}{1+s}}p\pi)$ are (strictly) isotropic conjugate points to the origin, they also are isotropic in (W^7, g_s). It proves (i).

For $\theta \in]0, \pi/2]$, if u is orthogonal to u_{0s}, one gets from (4.19) that $[[u, [u, v]]_{\mathfrak{u}^\bullet(2)}, u]$ is collinear to $[u, v]$ and $\rho = \frac{\sin^2\theta}{1+s}$. Then (ii) and (iii) follow using Theorem 2.9 (ii) (B) and Lemma 4.1 together with the fact that the base space of the canonical homogeneous Riemannian fibration is isometric to $\mathbb{C}P^2(1)$. □

Proof of Theorem 1.1. It follows using Propositions 3.2 and 3.4 and Lemma 4.12.
Proof of the Chavel's conjecture. From Theorems 4.8, 4.14 and 4.16, one obtains, for the cases (ii)-(iv), (vi) and (vii) in Theorem 1.1, the existence of non-isotropically conjugate points to the origin along any horizontal geodesic. Moreover, for the Berger space B^7, I. Chavel [6] finds non-isotropic conjugate points to the origin.

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